Answers to Problem Set 1

1 Question 1

(a) See attached graphs.

The value of the first function, $g(x)$, is zero when $x = 0$ and $x = 1$, and the function reaches its maximum at $x = 0.5$.

We can find a relative maximum and minimum for the second function in the interval when $0 \leq x \leq 10$.

The third function h(x), reaches its maximum at $f(0.25) = 0.25$, and cuts through the x-axis at $x = 0$ and $x = 1$.

(b)

$$g'(x) = \frac{\partial g(x)}{\partial x} = 1 - 2x,$$

$$f'(x) = \frac{\partial f(x)}{\partial x} = 3x^2 - 24x + 36, \quad \text{and}$$

$$h'(x) = \frac{\partial h(x)}{\partial x} = \frac{1}{2}x^{-\frac{1}{2}} - 1 = \frac{1}{2\sqrt{x}} - 1.$$

(c)

$$g'(x) = 0 \quad \Longrightarrow \quad 1 - 2x = 0 \quad \Longrightarrow \quad x = \frac{1}{2}, \quad \text{and}$$

$$g''(x) = -2 < 0.$$

Hence, $g(x)$ reaches it's maximum when $x = 0.5$.

$$f'(x) = 0 \quad \Longrightarrow \quad 3x^2 - 24x + 36 = 0 \quad \Longrightarrow \quad x_{1,2} = \frac{24 \pm \sqrt{24^2 - 4 \cdot 3 \cdot 36}}{2 \cdot 3}$$
$$\Longrightarrow \quad x_1 = 6 \quad \text{and} \quad x_2 = 2.$$

Since $f'(2) = f'(6) = 0$ these are the critical values for optima. As you can see in the picture, these are *relative* minimum and maximum.

Since $f'(x) > 0$ for $x < 2$ and $f'(x) < 0$ for $x > 2$ in the immediate neighborhood of $x = 2$, the corresponding value of the function $f(2) = 40$ is a relative maximum.

Similarly, since $f'(x) < 0$ for $x < 6$ and $f'(x) > 0$ for $x > 6$ in the immediate neighborhood of $x = 6$, the corresponding value of the function $f(6) = 8$ must be a relative minimum.

$$h'(x) = 0 \quad \Longrightarrow \quad \frac{1}{2}x^{-\frac{1}{2}} - 1 = 0 \quad \Longrightarrow \quad x = \frac{1}{4}, \quad \text{and}$$

$$h''(x) = -\frac{1}{2}x^{-\frac{3}{2}} < 0 \quad \text{for all} \quad x > 0.$$

Hence, $h(x)$ reaches it's maximum when $x = 0.25$.

2 Question 2

(a)

$$\frac{\partial f(x,y)}{\partial x} = 2x + y \quad \text{and}$$

$$\frac{\partial f(x,y)}{\partial y} = x + 4y + 3$$

(b)

$$\begin{cases} 2x + y & = 0 \\ x + 4y + 3 & = 0 \end{cases}$$

(c) ¿From the first equation we get $y = -2x$, and if we substitute this for y in the second equation we have

$$x + 4(-2x) + 3 = 0 \quad \Longrightarrow \quad x = \frac{3}{7}.$$

Using the value of x we can solve for y as

$$y = -2x = -2(\frac{3}{7}) = -\frac{6}{7}.$$

To determine whether $f(\frac{3}{7}, -\frac{6}{7}) = -1\frac{2}{7}$ is a maximum or minimum, we can either graph the function and study its values at points close to the optimum, or we can use the second derivative rule presented in class, and look at the second derivatives and cross derivatives of the function.

$$f_{xx} = \frac{\partial^2 f(x,y)}{\partial x^2} = 2, \qquad f_{xx} = \frac{\partial^2 f(x,y)}{\partial y^2} = 4,$$
$$f_{xy} = \frac{\partial^2 f(x,y)}{\partial x \partial y} = 1 \quad \text{and} \quad f_{yx} = \frac{\partial^2 f(x,y)}{\partial x \partial y} = 1$$

Since

$$f_{xx}, f_{yy} > 0 \quad \text{and}$$
$$f_{xx} \cdot f_{yy} > f_{xy}^2 \quad \text{or} \quad 2 \cdot 4 > 1$$

this is a minimum.

3 Question 3

(a)

$$L = -x_1^2 + 2x_1 - x_2^2 + 4x_2 - 5 - \lambda(x_1 + x_2 - 1)$$

(b)

$$\frac{\partial L}{\partial x_1} = -2x_1 + 2 - \lambda$$
$$\frac{\partial L}{\partial x_2} = -2x_2 + 4 - \lambda$$
$$\frac{\partial L}{\partial x_1} = -x_1 - x_2 + 1$$

(c)

$$\begin{cases} -2x_1 + 2 - \lambda & = 0 \\ -2x_2 + 4 - \lambda & = 0 \\ x_1 + x_2 - 1 & = 0 \end{cases}$$

¿From the first equation, we get $\lambda = 2 - 2x_1$. Using that in the second equation gives $-2x_2 + 4 = 2 - 2x_1 \implies x_1 = x_2 - 1$. If we substitute this into the third equation we get $x_2 - 1 + x_2 - 1 = 0 \implies 2x_2 = 2 \implies x_2 = 1$. And now, $x_1 = x_2 - 1 = 0$.

4 Question 4

(a) instead of building a Lagrangian for this constrained optimization problem, we can solve it simply by substituting for one of the variables in the objective function. Using the constraint, we can solve for $x_2 = 1 - x_1$. Substituting this for x_2 in the objective function gives us

$$\max_{x_1} -x_1^2 + 2x_1 - (1 - x_1)^2 + 4(1 - x_1) + 5.$$

(b) To find an optimum, we need to look at the tangent of the function, i.e.

$$\frac{\partial f(x_1)}{\partial x_1} = -2x_1 + 2 + 2(1 - x_1) - 4.$$

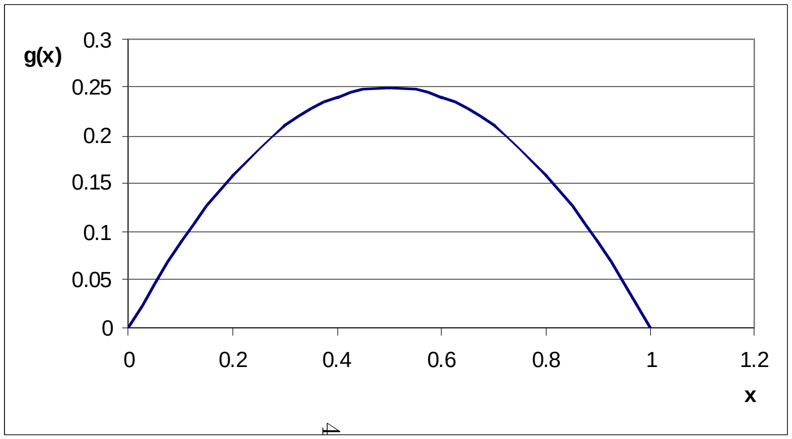
To find the maximum, we need to set $f'(x_1) = 0$. This gives us

$$-2x_1 + 2 + 2(1 - x_1) - 4 = 0 \implies -4x_1 = 0 \implies x_1 = 0.$$

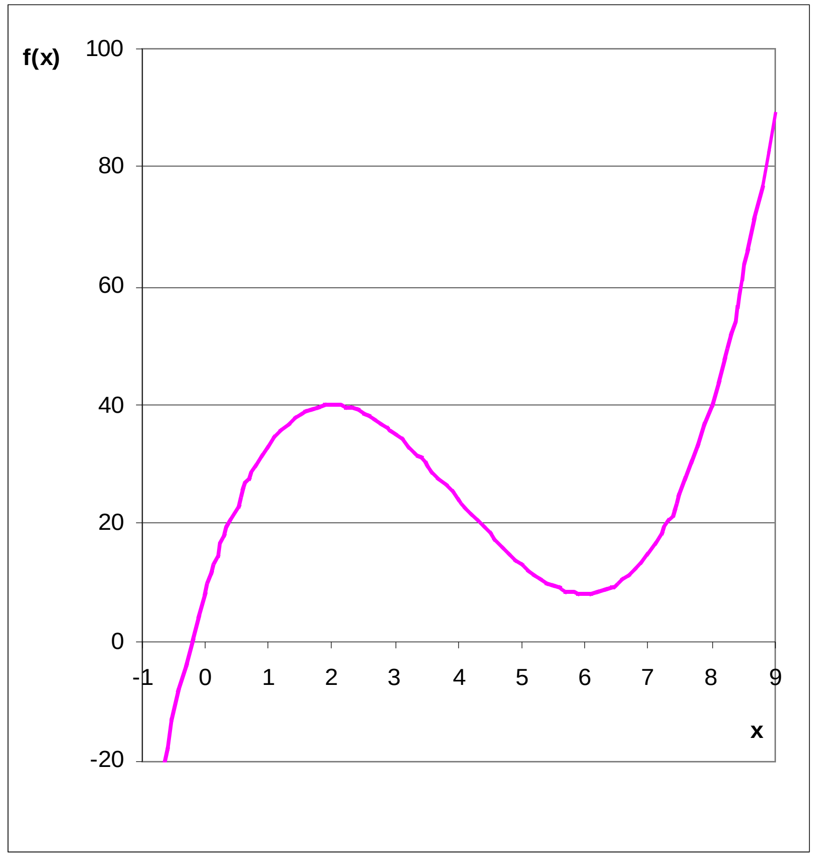
¿From (a) we know that $x_2 = 1 - x_1$, hence $x_2 = 1 - 0 = 1$. Also, we can see that this is a maximum, since $f_{x_1,x_1} = -4 < 0$.

(c) The solution $(x_1 = 0, x_2 = 1)$ is equal to the solution in question (3).

g(x) = x – x^2



f(x) = x^3 – 12x^2 + 36x + 8



h(x) = x^.5 – x