

Handout 3 and 4: Hints

Exercises H3 a)

(This is one solution among many!)

$$\begin{aligned} \text{Max } U(x, y) &= \sqrt{x - \alpha} \sqrt{x - \beta} \\ \text{s.t. } I &= p_x x + p_y y \end{aligned}$$

Set the Lagrange

$$\mathcal{L} = \sqrt{x - \alpha} \sqrt{x - \beta} - \lambda(p_x x + p_y y - I)$$

and derive with respect to x, y, and I. The MRS is as follow:

$$\frac{y - \beta}{x - \alpha} = \frac{p_x}{p_y}$$

Substitute in the Budget constraint to obtain uncompensated demand functions D_x and D_y :

$$\begin{aligned} D_x(p_x, p_y, I) &= \frac{I + p_x \alpha - p_y \beta}{2p_x} \\ D_y(p_x, p_y, I) &= \frac{I + p_y \beta - p_x \alpha}{2p_y} \end{aligned}$$

Plug these uncompensated demands into the Utility function to get the Indirect Utility function V:

$$\begin{aligned} V(p_x, p_y, I) &= \sqrt{\frac{I + p_x \alpha - p_y \beta}{2p_x} - \alpha} \sqrt{\frac{I + p_y \beta - p_x \alpha}{2p_y} - \beta} \\ V(p_x, p_y, I) &= \frac{I - p_x \alpha - p_y \beta}{2\sqrt{p_x p_y}} \end{aligned}$$

Reverse to obtain the Expenditure function E:

$$E(p_x, p_y, U) = 2U\sqrt{p_x p_y} + p_x \alpha + p_y \beta$$

Use the Shephard Lemma to derive the compensated demand functions D_x^c and D_y^c :

$$\begin{aligned} \frac{\partial E(\cdot)}{\partial p_x} &= U\sqrt{\frac{p_y}{p_x}} + \alpha = D_x^c(p_x, p_y, U) \\ \frac{\partial E(\cdot)}{\partial p_y} &= U\sqrt{\frac{p_x}{p_y}} + \beta = D_y^c(p_x, p_y, U) \end{aligned}$$

α and β are the minimal consumption amount that must be bought by the consumer.

Exercise H3 b)

Marshallian (uncompensated) demand function $D_i(p, E(p, \bar{U}))$

Hicksian (compensated) demand function $D_i^c(p, \bar{U})$

$$D_i(p, E(p, \bar{U})) = D_i^c(p, \bar{U})$$

Differentiate the above equation with respect to p_i :

$$\frac{\partial D_i}{\partial p_i} + \frac{\partial E(\cdot)}{\partial p_i} \frac{\partial D_i}{\partial I} = \frac{\partial D_i^c}{\partial p_i}$$

By the Shephard Lemma, we know that $\frac{\partial E(\cdot)}{\partial p_i} = x_i$, then:

$$\frac{\partial D_i}{\partial p_i} = \frac{\partial D_i^c}{\partial p_i} - x_i \frac{\partial D_i}{\partial I}$$

The first RHS term corresponds to the substitution effect while the second RHS to the income effect. The substitution effect is always negative because $\frac{\partial D_i^c}{\partial p_i}$ is the second derivative of the expenditure function which is concave. The income effect can be positive (normal good) or negative (inferior good).

Total Effect	Substitution Effect	Income Effect	Type of good
$\frac{\partial D_i}{\partial p_i}$	$= \frac{\partial D_i^c}{\partial p_i}$	$- x_i \frac{\partial D_i}{\partial I}$	
(-)	(-)	(+)	Normal
(-)	(-)	(-)	Inferior
(+)	(-)	(-)	Giffen

Exercise H4 1)

1. a.

$$\text{Max } U(x, y) = x_1^\alpha x_2^\beta x_3^\gamma$$

$$\text{st : } I = p_1 x_1 + p_2 x_2 + p_3 x_3$$

Set the Lagrange

$$\mathcal{L} = x_1^\alpha x_2^\beta x_3^\gamma - \lambda(p_1 x_1 + p_2 x_2 + p_3 x_3 - I)$$

and derive the uncompensated demand function

D_1, D_2, D_3 :

$$D_1(p_1, p_2, p_3, I) = \frac{\alpha I}{p_1}$$

$$D_2(p_1, p_2, p_3, I) = \frac{\beta I}{p_3}$$

$$D_3(p_1, p_2, p_3, I) = \frac{\gamma I}{p_3}$$

Plug these uncompensated demands into the Utility function to get the Indirect Utility function V:

$$V(p_1, p_2, p_3, I) = \left(\frac{\alpha I}{p_1}\right)^\alpha \left(\frac{\beta I}{p_2}\right)^\beta \left(\frac{\gamma I}{p_3}\right)^\gamma$$

$$V(p_1, p_2, p_3, I) = I \left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{\beta}{p_2}\right)^\beta \left(\frac{\gamma}{p_3}\right)^\gamma \text{ with } \alpha + \beta + \gamma = 1$$

Reverse to obtain the Expenditure function E:

$$E(p_1, p_2, p_3, U) = \frac{U}{\left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{\beta}{p_2}\right)^\beta \left(\frac{\gamma}{p_3}\right)^\gamma}$$

b. $U(x_1, x_2) = \min(x_1, x_2)$. Perfect complement

$$I = p_1 x_1 + p_2 x_2$$

$$D_1(p_1, p_2, I) = \frac{I}{p_1 + p_2}$$

$$D_2(p_1, p_2, I) = \frac{I}{p_1 + p_2}$$

Plug these uncompensated demands into the Utility function to get the Indirect Utility function V:

$$V(p_1, p_2, I) = \frac{I}{p_1 + p_2}$$

Reverse to obtain the Expenditure function E:

$$E(p_1, p_2, U) = U(p_1 + p_2)$$

c. $U(x_1, x_2) = x_1 + x_2$. Perfect substitute, there are 3 cases:

$$I = p_1 x_1 + p_2 x_2$$

$$p_1 > p_2 \Rightarrow D_1(p_1, p_2, I) = 0; D_2(p_1, p_2, I) = \frac{I}{p_2}$$

$$p_1 = p_2 \Rightarrow D_1(p_1, p_2, I) = D_2(p_1, p_2, I) = \frac{I}{p_2} = \frac{I}{p_1}$$

$$p_1 < p_2 \Rightarrow D_1(p_1, p_2, I) = \frac{I}{p_1}; D_2(p_1, p_2, I) = 0$$