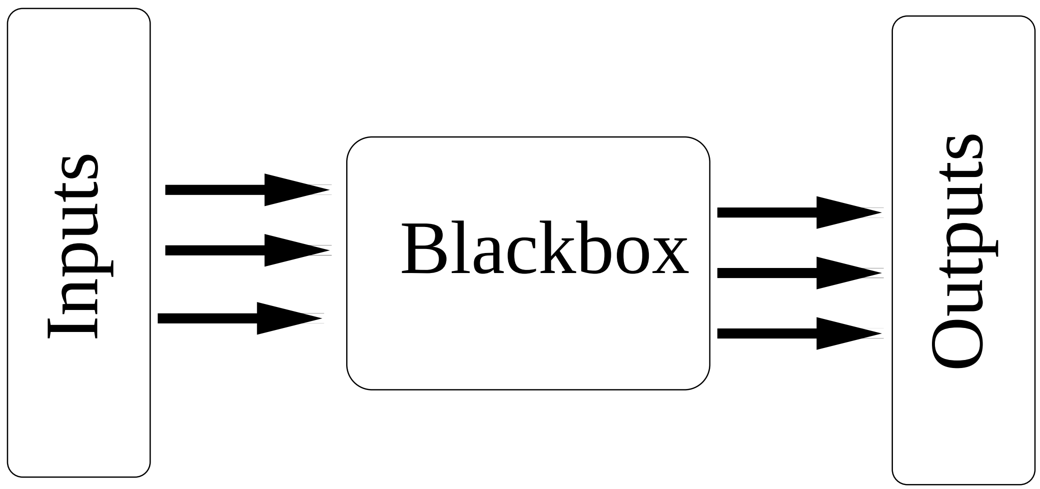
# *Lecture 1: Theory of the firm*

- Technology
- Cost minimization
- Profit maximization

Let us briefly mention one interesting issue before we start: why are there companies, i.e. many individuals working together in one organization?
Exciting question which we postpone for a while — suspense, suspense.
For the time being we take the existence of companies as given and try to formalize their behavior (cost minimization or more generally profit maximization) and the technology they face.

# Technology

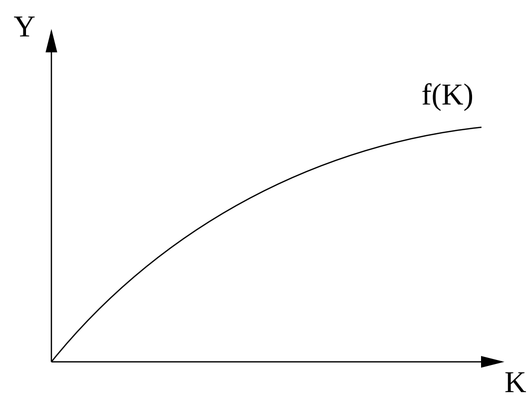
Technology — the hard facts of life companies face. It hasn't much to do with economics (rather engineering) but we need a formalization or description. Think of it as a black box converting inputs into outputs.




Since more than one output would be hard to handle, we will always assume that companies produce only one output (or can be subdivided into units that do). That is, we abstract from economies of scope (= synergies of producing several things together).

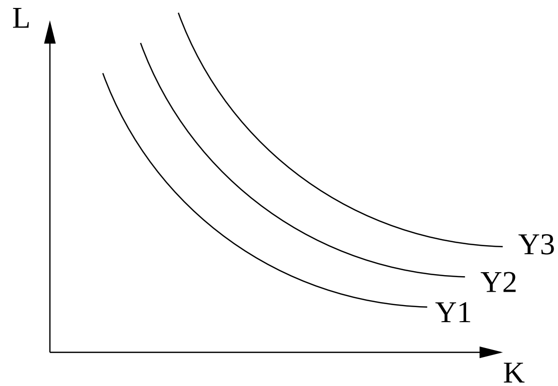
With only one output, we can formalize the production process as a function $Y = f(K, L, ...)$ where $Y$ is output quantity and $K, L, ...$ are input quantities.

With only one input this is easy to depict graphically:




But with more than one input 2-dimensional graphics become tricky.
For example, with 3 inputs we would have 4 dimensions (3 inputs plus 1 output) which would hardly fit onto a two-dimensional page.

The case of 2 inputs is still manageable, however.




Instead of the 3-dimensional production function (2 inputs $+$ 1 output $=$ 3) we depict production isoquants which are defined by $\bar{Y} = f(K, L)$ for different $\bar{Y}$s. Same trick or concept as altitude lines on geographical maps.

# Characteristics of technology

- **Marginal Product:** how does output change if we increase **one** input marginally. mathematically:

$$MP_K = \frac{\delta f(K, L, ...)}{\delta K}$$
$$MP_L = \frac{\delta f(K, L, ...)}{\delta L}$$

  If you think of the 2-input production function as a mountain this is its slope in **one** direction.
  The marginal product will usually be positive though decreasingly so. Diminishing MP is the standard assumption.
  In the extreme, it could become negative if you send the 1000th farmhand onto a field he/she might reduce output.

- **Returns to scale:** how does output change if we increase **all** inputs marginally, in other words if we scale up production.

mathematically (for homogeneous production functions):

$$f(\lambda K, \lambda L, \lambda ...) \quad >? <? =? \quad \lambda f(K, L, ...)$$

If the LHS $>$ RHS, we speak of increasing returns to scale (IRS), equality corresponds to constant returns to scale (CRS), and the case of LHS $<$ RHS is called decreasing returns to scale (DRS).
Again in terms of our 2-input production mountain, this tells us the slope when we climb up diagonally (increasing both inputs) — proportional to the increase in inputs, or more/less than proportional. Think about how this concept relates to the distances between isoquants!

- **"free disposal"**: more inputs cannot reduce output (one could throw inputs away or dispose of them, after all)

- **"no free lunch"**: no (positive) output without inputs

# Cost Minimization

We assume/consider a company that wants to achieve the following objective: produce a **given** output quantity at minimal cost.

The company is assumed to be a small player and takes prices (for inputs as well as for output) as given — no market power on either side.

Note that with one input this is really easy: the production function tells us how much of the one and only input we need, and at a given factor price this input quantity has its cost. Not much of a minimization.

The problem becomes interesting only if production requires at least two inputs. In fact, for simplicity, we will mostly limit ourselves to the case of two inputs. Then the problem is finding the cheapest input mix resulting in the required output.

**Mathematically:**

$$\min_{K,L} \quad rK + wL \qquad \text{(that's the objective)}$$

subject to (s.t. from now on):

$$\bar{Y} = f(K, L) \qquad \text{(constrained by the hard facts of life)}$$

We solve this constrained optimization problem by setting up the Lagrangean:

$$\mathcal{L} = rK + wL + \lambda(\bar{Y} - f(K, L))$$

these are the problem's first order conditions (FOCs):

$$\frac{\delta \mathcal{L}}{\delta K} = r - \lambda f_K = 0$$

$$\frac{\delta \mathcal{L}}{\delta L} = w - \lambda f_L = 0$$

dividing one by the other gives:

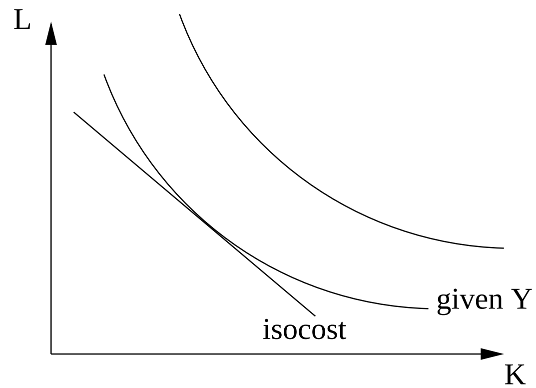
$$\frac{r}{w} = \frac{f_K}{f_L} \equiv MRTS$$

where MRTS is shorthand for the marginal rate of technical substitution.

Note that we have two equations — the constraint as well as relative input price equals MRTS — in two unknows (K and L). This system can be solved to obtain the so-called **conditional** (b/c they are conditional on the output quantity) **factor demands**:

$$K(w, r, \bar{Y}) \text{ and } L(w, r, \bar{Y})$$

**Graphically:**




The slope of the isoquants is the MRTS (totally differentiating $\bar{Y} = f(K, L)$ will tell you that) while the slope of the isocost lines is just the relative input price $r/w$ (again totally differentiate its definition, $\bar{C} = rK + wL$, to see this).

Plug the conditional factor demands back into the definition of cost ($rK + wL$) to obtain what is called the (minimal) **cost function**:

$$C(w, r, \bar{Y}) \equiv rK(w, r, \bar{Y}) + wL(w, r, \bar{Y})$$

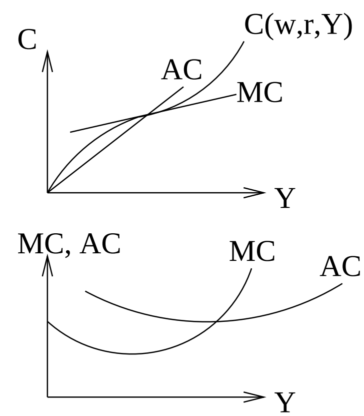
We have now solved the cost minimization problem. Its solution is the cheapest input combination ($K(w, r, \bar{Y}$ and $L(w, r, \bar{Y})$ and the cost function tells us what the cheapest way to produce a given output quantity actually costs.

A few more things about cost:

- first two definitions:

$$\textbf{(marginal cost)} MC \equiv \frac{\delta C}{\delta Y}$$

$$\textbf{(average cost)} AC \equiv \frac{C}{Y}$$




To show that MC must intersect AC in the latter's minimum note that said minimum is characterized by $\delta AC/\delta Y = 0$. Using the definition of $AC$ ($\equiv C(Y)/Y$) and explicitely calculating the first order condition should lead you to the desired result. Intuitively, $MC$ pulls down $AC$ as long as its below, and pulls it up once it's above.

- **Characteristics of the cost function:**
  -

$$\frac{\delta C(w,r,\bar{Y})}{\delta w} = L(w,r,\bar{Y})$$

$$\frac{\delta C(w,r,\bar{Y})}{\delta r} = K(w,r,\bar{Y})$$

  due to the envelope theorem
  - linear homogeneous in factor prices
  - concave in prices

- **Relationship between $MC$ and returns to scale:**
  - CRS corresponds to constant $MC$
  - IRS corresponds to decreasing $MC$
  - DRS corresponds to increasing $MC$
  - relate $C(Y = 1)$ vs. $C(Y = 2)$ vs. $C(Y = 3)$ to the changing distances between isoquants to see this.

# Profit Maximization

Output is no longer given. We assume that a company chooses inputs and output in order to maximize profits. This is the standard assumption regarding firms' behavior. There are alternatives — say maximizing sales b/c that maximizes a CEO's prestige — but one can show that if some companies maximize profits while others do "strange" things the former will dominate and the others won't survive.

How does profit maximization relate to cost minimization: we will show below that the former implies the latter but not vice versa. That is, a profit maximizing firm always minimizes costs (how else could the profit maximizing output be really profit maximizing) but a cost minimizer does not necessarily maximize profits (simply because the given output quantity might not be the profit maximizing one).

**Mathematically:**

$$\max_{K,L,Y} \quad \text{profit} \quad \pi = pY - rK - wL$$

s.t.

$$Y = f(K, L)$$

There are (at least) three ways of solving this optimization:

1. plain Lagrange

2. substitute the constraint into the objective

3. short-cut via cost minimization

1. Forget the first way. It is possible but unnecessarily complicated. Lagrange adds one dimension to the problem due to the multiplier. You would end up with a constrained maximization over inputs, output, and the multiplier — unnecessarily painful.

2. The second way by contrast reduces the problem by one dimension (we substitute the constraint into the objective and thereby get rid of output) and in addition leads to an unconstrained maximization. Let's briefly see how it works for 2 inputs (because this will prove that profit max implies cost min) and then make life even easier and do it for only one input.

$$\max_{K,L} \pi = pf(K,L) - rK - wL$$

FOCs:

$$K : \frac{\delta\pi}{\delta K} = pf_K - r = 0$$
$$L : \frac{\delta\pi}{\delta L} = pf_L - w = 0$$

Dividing one by the other gives you the same tangency condition (relative input price equals MRTS) as we got when doing cost minimization. Therefore profit max implies cost min. To actually solve the optimization we would not divide one FOC by the other (because it reduces a system of 2 eqtns in two unknowns (K, L) to one eqtn in two unknowns — dead end) but instead solve the 2 FOCs for K and L. Pls continue on your own.

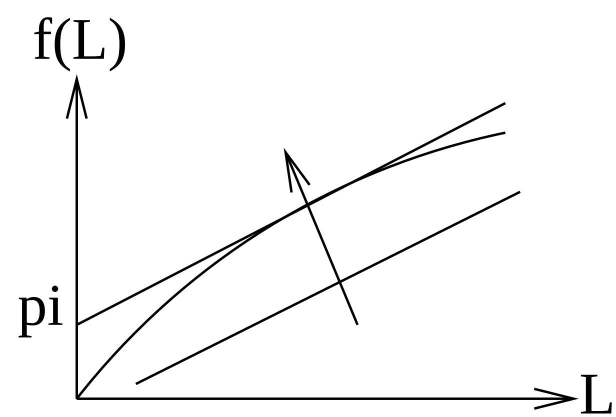
Now, let's do it for one input:

$$\max_L \pi = pY - wL = pf(L) - wL$$

FOC:

$$pf_L - w = 0 \text{ or } MP_L = w/p$$

This can conceptionally be solved for the unconditional factor demand $L(w,p)$. In other words, it can be solved once we assume a specific prod fct. Then plug $L(w,p)$ into the prod fct to get the output supply function $Y(w,p)$. Note that both these functions will in general be functions of the whole price vector, $(w,p)$ in this case.



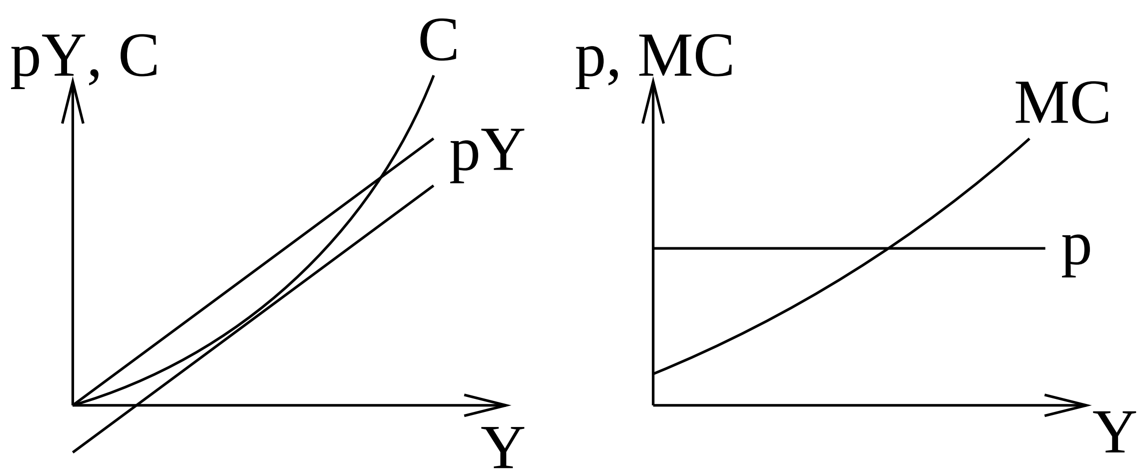
3. Short-cut via cost minimization. Note that here we use what we have proved above, namely that profit max implies cost min. The number of inputs doesn't really matter for this approach so let's do it for 2 inputs.

$$max_Y \pi = pY - C(w, r, Y)$$

FOC:

$$p - MC = 0$$

graphically:



Note that, as above, only a specific prod fct would allow us to actually solve for output supply and unconditional factor demands. A specific prod fct would give rise to a specific cost fct and a specific form of marginal cost. We could then solve the FOC for $Y(p, w, r)$. Finally, plug this output supply function into the conditional factor demands (obtained from the cost minimization problem we already solved) in order to obtain the unconditional factor demand functions $L(w, r, p)$ and $K(w, r, p)$. Note again that in general these fcts will depend on the whole price vector.

No matter which way we choose, we obtain the output supply function $Y(p, w, r)$ and the unconditional factor demands $K(p, w, r)$, $L(p, w, r)$, etc. as functions of the price vector. We can plug these functions into the definition of profit, i.e. into $\pi = pY - rK - wL$ to obtain what is called the (maximized) **profit function**:

$$\pi(p, w, r) \equiv pY(p, w, r) - rK(p, w, r) - wL(p, w, r)$$

This function tells us what maximal profit the company will achieve given the price vector if it chooses inputs and (consequently) output optimally.

A few more things about profit:

- Hotelling's Lemma:

$$\frac{\delta\pi(p,w,r)}{\delta w} = L(p,w,r)$$

similarly for other factor(s)

$$\frac{\delta\pi(p,w,r)}{\delta p} = Y(p,w,r)$$

Again this can be proved applying the envelope theorem.

- the profit function is convex in prices

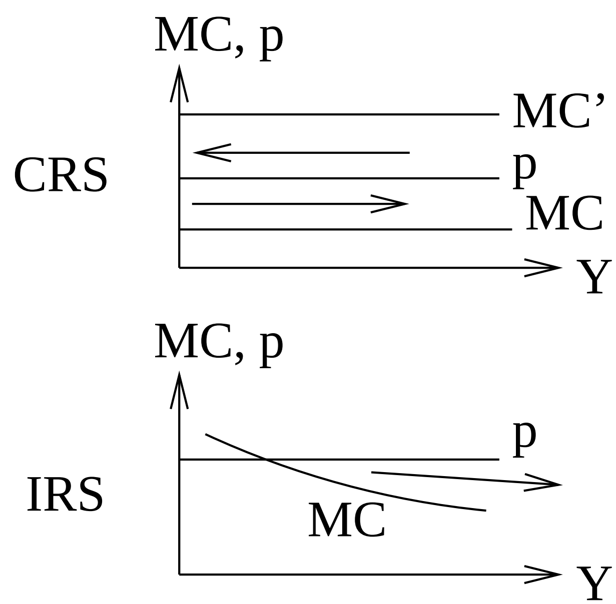
- it is linear-homogeneous in prices

- Choosing the third way, the second order (sufficient) condition would have been (always check:)

$$\frac{\delta^2 C(Y)}{\delta Y^2} > 0$$

In other words, increasing marginal cost or DRS!

So what about IRS or CRS?




Not well defined — either no production or off to infinity.